J. Fluid Mech. (2005), *vol.* 540, *pp.* 175–187. © 2005 Cambridge University Press
doi:10.1017/S0022112005005835 Printed in the United Kingdom

On deflectors of optimum shape

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(Received 25 October 2004 and in revised form 18 April 2005)

In this paper the problem of the jet deflector of optimum shape has been solved. The deflector divides a jet that effuses from a semi-infinite channel of finite width. The goal of the investigation is to define the shape of the deflector that provides either its minimum wetted arclength under the given deflection angle or (which is equivalent) the deflection of the jet through the maximum angle under the given arclength of the deflector. An exact analytical solution of the problem has been found and it has been shown that the solution realizes a global extreme. A series of optimum deflectors is constructed for a variety of deflection angles and contraction jet coefficients.

1. Introduction

In the theory of jets and cavities there are only few results related to finding optimal shapes and the corresponding exact bounds on hydrodynamic forces exerted on curved obstacles. The first problem of this kind was solved by Lavrentieff (1938), who determined the shape of minimum drag in Helmholtz flow for a two-dimensional symmetrical body which is constrained to lie within a given rectangle. Later Serrin (1953) extended the results of Lavrentieff to the axially symmetric case.

Wu & Whitney (1972) applied the technique of variational calculus to determine the shape of a curved plate planing on a free surface and creating maximal lift, the wetted arclength and the chord length of the plate being fixed. As a continuation of this work Whitney (1972) considered a minimum drag problem in Helmholtz flow under the same isoperimetric restriction as in Wu & Whitney (1972). No exact bounds on hydrodynamic forces but only approximate solutions were obtained by Wu & Whitney (1972), Whitney (1972).

In Maklakov (1988), Maklakov & Uglov (1995), Maklakov (1999) and Maklakov (2004) a series of problems on defining optimum hydrodynamic shapes in free-surface flows has been studied. The problems have been reduced to maximization of nonlinear functionals under nonlinear restrictions. The global maximum has been found by means of non-trivial application of Jensen's inequality. In Maklakov (1988) the deflectors of optimum shape that separate a free jet have been determined. In Maklakov & Uglov (1995) the problem of the ideal optimum parachute for the Joukovsky–Roshko–Eppler wake model has been solved. In Maklakov (1999) the optimum shape of a planing plate has been found. In Maklakov (2004) the problem of the optimum parachute in Helmholtz flow has been discussed. For all of the above problems exact analytical solutions have been constructed. A systematical presentation of the results can be found in the monograph by Maklakov (1997).

In this paper the problem solved in Maklakov (1988) is essentially generalized. The deflector divides not a free jet but a jet that issues from a semi-infinite channel of finite width. Deflectors have applications in many technical domains. In particular,

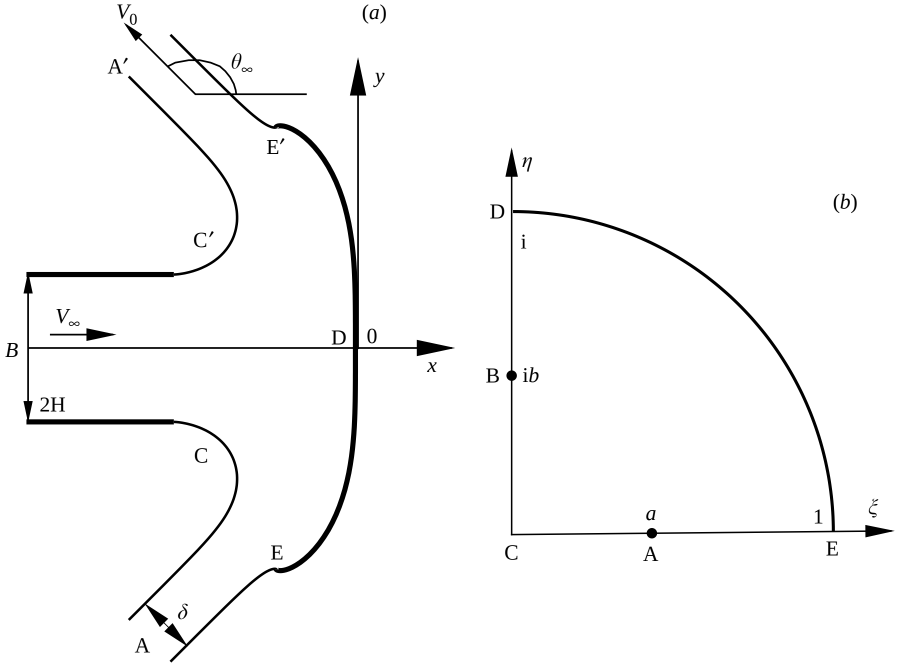

FIGURE 1. Flow in (a) the physical z-plane, (b) parametric t-plane.

the thrust reversal device of bucket type for turbojet engines can be considered as a deflector that turns the jet to create the backthrust that needs for braking an aircraft. The principal quantity that defines the effectiveness of the deflector is the deflection angle of the jet. The goal of the investigation carried out in the work is to define the shape of the deflector that provides either its minimum wetted arclength under the given deflection angle or (which is equivalent) the deflection of the jet through the maximum angle under the given arclength of the deflector. In the latter formulation the maximum deflection angle creates the maximum drag force (the maximum backthrust). The location of the deflector with respect to the channel is uniquely determined as part of the solution process. An important feature of the problem is that the contraction coefficient of the jet is assumed to be given. A series of optimum deflectors is constructed for a variety of deflection angles and contraction coefficients. It is shown that the solutions obtained realize global extremes.

2. Problem formulation

Let us consider a two-dimensional jet of a fluid of density ρ that effuses from a semi-infinite channel CBC′ of width $2H$. The fluid is ideal and incompressible. The flow velocity V_∞ at infinity in the channel is assumed to be so large that the effect of gravity may be neglected. The jet is deflected by the curved plate EDE′ (deflector) through the angle θ_∞. The width of the jet at the points A and A′ at infinity is δ. The deflector as well as the entire flow are symmetric with respect to the axis of symmetry of the channel. The x-axis is directed along the axis of symmetry, the y-axis is directed vertically upward and goes through the stagnation point D of the flow (see figure 1a).

We shall call the ratio $k=\delta/H$ the contraction coefficient of the jet. This ratio is assumed to change in the range $0<k\leqslant 1$. We suppose that the parameters ρ, H, V_∞,

θ_∞ and k are given, the inclination angle θ_∞ being in the range $0 \leqslant \theta_\infty \leqslant \pi$. If θ_∞ is beyond of this range, then the jets CAE and C′A′E′ will intersect each other and the flow loses its physical meaning. The problem is to find the shape of the deflector and its location with respect to the channel (the distance d from the point D to the end channel section CC′) so that the arclength $2L$ of the deflector is minimal.

It is to be noted that specifying the contraction coefficient k allows us to define the constant velocity V_0 along the free streamlines. Comparing the fluxes through the channel and the jets we find that

$$\frac{V_0}{V_\infty} = \frac{1}{k}. \tag{2.1}$$

Moreover, in specifying k we also specify the pressure difference between the inlet of the channel and the atmosphere. Indeed, from Bernoulli's equation it follows that the pressure difference coefficient

$$Q = \frac{p_\infty - p_0}{\rho V_\infty^2/2} = \frac{V_0^2}{V_\infty^2} - 1 = \frac{1}{k^2} - 1,$$

where p_∞ is the pressure at the point B at infinity of the channel, p_0 is the atmospheric pressure.

In accordance with Gurevich (1965) and (2.1) the drag force D of the deflector can be determined by the equation

$$D = \rho H V_\infty^2 \left(\frac{1}{k^2} - \frac{2}{k}\cos\theta_\infty + 1 \right).$$

So, given the parameters $\rho, H, V_\infty, \theta_\infty$ and k the drag force D is given too. If we treat the problem as that of a thrust reversal device of bucket type, then D is the backthrust, which is known in advance. Minimizing the arclength of the deflector means minimizing the weight of the device. Later we shall show that the optimum shapes found provide the maximum of the inclination angle θ_∞ (the maximum of the drag force D) for given parameters ρ, H, V_∞, k and the arclength of the deflector $2L$.

By virtue of symmetry we consider only the lower half of the flow in the physical plane $z = x + \mathrm{i}\, y$ and map this half onto a quarter of the unit disk in the parametric plane $t = \xi + \mathrm{i}\,\eta$. The correspondence of points can be seen in figure 1(*a,b*). Let W be the complex potential of the flow. The derivative $\mathrm{d}W/\mathrm{d}t$ of the complex potential can be found by Chaplygin's singular point method (see Gurevich 1965):

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \phi_0 \frac{t(t^4-1)}{(t^2-a^2)(1-a^2t^2)(t^2+b^2)(1+b^2t^2)}, \tag{2.2}$$

where ϕ_0 is a constant that has the dimension of the velocity potential, a and $\mathrm{i}b$ are the images of the points A and B in the t-plane respectively.

We introduce the logarithmic hodograph variable

$$\omega(t) = \log \frac{V_0\, \mathrm{d}z}{\mathrm{d}W} = \log \frac{V_0}{V} + \mathrm{i}\theta, \tag{2.3}$$

where V is the flow velocity, θ is the inclination of the velocity vector.

Let us assume that the function

$$\nu(\sigma) = \operatorname{Re}\omega(\mathrm{e}^{\mathrm{i}\sigma}), \qquad 0 \leqslant \sigma \leqslant \pi/2 \tag{2.4}$$

is known.

On the free streamline CAE we have

$$\operatorname{Re}\omega(\xi) = 0, \qquad 0 \leqslant \xi \leqslant 1. \tag{2.5}$$

On the solid horizontal wall BC and the axis of symmetry BD we have

$$\operatorname{Im}\omega(\mathrm{i}\,\eta) = 0, \qquad 0 \leqslant \eta \leqslant 1. \tag{2.6}$$

According to (2.5), (2.6) the function $\omega(t)$ is imaginary on the real axis and real on the imaginary axis; therefore it can be continued analytically across the segments CE and CD onto the entire unit disk. With the Schwarz–Poisson formula we obtain from (2.4)–(2.6) that

$$\omega(t) = -\frac{4\,\mathrm{i}\,t}{\pi}(t^2+1)\int_0^{\pi/2} \frac{\nu(\sigma)\sin\sigma\,\mathrm{d}\sigma}{t^4+1-2t^2\cos 2\sigma}. \tag{2.7}$$

Calculating the residue of the function (2.2) at the point $t = \mathrm{i}b$ and taking into account (2.1) we get

$$HV_0 = \frac{\phi_0\pi}{2k(a^2+b^2)(1+a^2b^2)}.$$

Now we can exclude ϕ_0 from equation (2.2) and express all features of the flow in terms of the function $\nu(\sigma)$ and the parameters a and b. In particular, the derivative of the conformal mapping $z = z(t)$ is determined as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{2Hk(a^2+b^2)(1+a^2b^2)t(t^4-1)\exp\omega(t)}{\pi(t^2-a^2)(1-a^2t^2)(t^2+b^2)(1+b^2t^2)}. \tag{2.8}$$

Because θ_∞ and $k = V_\infty/V_0$ are given we deduce from (2.3), (2.7) that the function $\nu(\sigma)$ and the parameters a, b must satisfy the following restrictions:

$$-\operatorname{Im}\omega(a) = \frac{4a(a^2+1)}{\pi}\int_0^{\pi/2} \frac{\nu(\sigma)\sin\sigma\,\mathrm{d}\sigma}{a^4+1-2a^2\cos 2\sigma} = \theta_\infty, \tag{2.9}$$

$$\operatorname{Re}\omega(\mathrm{i}\,b) = \frac{4b(1-b^2)}{\pi}\int_0^{\pi/2} \frac{\nu(\sigma)\sin\sigma\,\mathrm{d}\sigma}{b^4+1+2b^2\cos 2\sigma} = -\log k. \tag{2.10}$$

The arclength of the deflector is found from (2.8):

$$\begin{aligned}\frac{L}{H} &= J(\nu, a, b)\\ &= \frac{4k(a^2+b^2)(1+a^2b^2)}{\pi}\int_0^{\pi/2} \frac{\sin 2\sigma\exp[\nu(\sigma)]\,\mathrm{d}\sigma}{(a^4+1-2a^2\cos 2\sigma)(b^4+1+2b^2\cos 2\sigma)}.\end{aligned} \tag{2.11}$$

So, the problem of minimizing the wetted arclength $2L$ of the deflector for given ρ, H, V_∞, θ_∞ and k is equivalent to that of finding the parameters a, b in the ranges $0 < a < 1$, $0 < b < 1$ and the function $\nu(\sigma)$, defined in the range $0 \leqslant \sigma \leqslant \pi/2$, so that a, b and $\nu(\sigma)$ minimize the functional $J(\nu, a, b)$ under the restrictions (2.9), (2.10). Before proceeding to solving the problem (2.9)–(2.11) we consider two particular cases of it, namely to find the shape of the optimum deflector in a free jet and that inside an infinite channel. Although the first particular problem has already been solved in Maklakov (1988), we present its solution here so as not to lose the integrity of the text.

3. Optimum deflector in a free jet

Let us consider the deflector that divides a free jet. In this case the points B and C coincide, the contraction coefficient $k = 1$ and the mathematical parameter $b = 0$. It follows from (2.9)–(2.11) that at $k = 1$ and $b = 0$ we need to minimize the functional

$$\frac{L}{H} = J_1(\nu, a) = \frac{4a^2}{\pi} \int_0^{\pi/2} \frac{\sin 2\sigma \exp[\nu(\sigma)]\,\mathrm{d}\sigma}{a^4 + 1 - 2a^2 \cos 2\sigma} \quad \text{(min)} \tag{3.1}$$

under the restriction (2.9). We have here only one restriction because at $k = 1$ and $b = 0$ equation (2.10) degenerates to an identity.

To do the minimization we introduce a new unknown function

$$\lambda(\sigma) = \nu(\sigma) + \log\cos\sigma. \tag{3.2}$$

Substituting $\lambda(\sigma)$ for $\nu(\sigma)$ in (3.1), (2.9) we arrive at the problem:

$$J_1(\lambda, a) = \frac{8a^2}{\pi} \int_0^{\pi/2} \frac{\sin\sigma \exp[\nu(\sigma)]\,\mathrm{d}\sigma}{a^4 + 1 - 2a^2 \cos 2\sigma} \quad \text{(min)} \tag{3.3}$$

under the restriction

$$\frac{4a(a^2+1)}{\pi} \int_0^{\pi/2} \frac{\lambda(\sigma)\sin\sigma\,\mathrm{d}\sigma}{a^4 + 1 - 2a^2\cos 2\sigma} = \theta_\infty - I(a), \tag{3.4}$$

where

$$I(a) = -\frac{4a(a^2+1)}{\pi} \int_0^{\pi/2} \frac{\sin\sigma \log\cos\sigma\,\mathrm{d}\sigma}{a^4 + 1 - 2a^2\cos 2\sigma}. \tag{3.5}$$

The denominator of the integrand in (3.5) is represented as follows:

$$\Delta = a^4 + 1 - 2a^2\cos 2\sigma = (a^2+1)(1 - \alpha^2\cos^2\sigma),$$

where $\alpha = 2a/(a^2+1)$. Then

$$\Delta^{-1} = \frac{1}{a^2+1} \sum_{n=0}^{\infty} \alpha^{2n} \cos^{2n}\sigma.$$

After substitution of this expression in (3.5) and integration we obtain

$$I(a) = T\left(\frac{2a}{a^2+1}\right), \quad T(\alpha) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)^2}. \tag{3.6}$$

It is worth noting that

$$T(\alpha) = \frac{1}{\pi}[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(-\alpha)], \quad \mathrm{Li}_2(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^n}{n^2},$$

$\mathrm{Li}_2(\alpha)$ being the Euler dilogarithm (see Prudnikov, Brychkov & Marichev 1988).

The function $T(\alpha)$ appeared first in the paper by Maklakov & Uglov (1995), in which the shape of a curved plate of maximum drag in flow with a wake was determined. It follows from equation (3.6) that this function is analytic in the unit disk ($|\alpha| < 1$). In Maklakov & Uglov (1995) it has been demonstrated that $T(\alpha)$ can be continued analytically from the upper half of the unit disk to the upper half-plane by the formula

$$T(\alpha) = T(-1/\alpha) + \mathrm{i}\log(\alpha) + \pi/2, \qquad \operatorname{Im}\alpha > 0. \tag{3.7}$$

In what follows we shall use (3.7) to compute $T(\alpha)$ for $|\alpha| > 1$.

Let us assume that the function $\lambda(\sigma)$ and the parameter a satisfy the restriction (3.4). The functional (3.3) can be estimated from below by a special case of Jensen's inequality (see Hardy, Littlewood & Polya 1934, p. 138):

$$\int_p^q f(\sigma)\,\mathrm{e}^{\lambda(\sigma)}\,\mathrm{d}\sigma \geqslant \exp\left[\int_p^q f(\sigma)\,\lambda(\sigma)\,\mathrm{d}\sigma\right], \tag{3.8}$$

where

$$\int_p^q f(\sigma)\,d\sigma = 1, \qquad f(\sigma) \geqslant 0,$$

the equality in (3.8) being possible if and only if $\lambda(\sigma) = C \equiv$ const. Applying (3.8) to (3.1) with allowance made for (3.4), (3.6) we get the following estimate:

$$J_1(\nu, a) \geqslant G_1(a), \tag{3.9}$$

where

$$G_1(a) = \frac{8a^2}{\pi} g_1(a) \exp\left\{\frac{\pi}{4g_1(a)} \frac{\theta_\infty - T[2a/(a^2+1)]}{a(a^2+1)}\right\}, \tag{3.10}$$

$$g_1(a) = \int_0^{\pi/2} \frac{\sin\sigma\,\mathrm{d}\sigma}{a^4 + 1 - 2a^2\cos 2\sigma} = \frac{1}{2(a^2+1)a} \log\frac{1+a}{1-a}.$$

The estimate (3.9) is correct for any function $\nu(\sigma)$ and parameter a that satisfy the restriction (2.9), the equality in (3.9) being possible if and only if

$$\nu(\sigma) = C - \log\cos\sigma, \tag{3.11}$$

where C is a certain constant. Inserting (3.11) in (2.9) we find that for $\nu(\sigma)$ defined by (3.11) and

$$C = \frac{\pi}{2} \frac{\theta_\infty - T[2a/(a^2+1)]}{\log[(1+a)/(1-a)]} \tag{3.12}$$

the restriction (3.11) is fulfilled So, for the function (3.11) with C satisfying (3.12) the equality holds in the estimate (3.9). For any other $\nu(\sigma)$ that satisfy (2.9) we shall have in (3.9) a strict inequality.

Let us find a minimum of the function $G_1(a)$ for $0 < a < 1$. To do so we differentiate $G_1(a)$ taking into account that

$$\frac{\mathrm{d}T}{\mathrm{d}\alpha} = \frac{1}{\pi\alpha} \log\frac{1+\alpha}{1-\alpha}.$$

After a little algebra differentiation gives

$$\frac{\mathrm{d}G_1(a)}{\mathrm{d}a} = \frac{\pi G_1(a)}{1-a^2} \frac{\dfrac{2}{\pi}\log\dfrac{1+a}{1-a} + T\left(\dfrac{2a}{a^2+1}\right) - \theta_\infty}{\log^2[(1+a)/(1-a)]}. \tag{3.13}$$

It follows from (3.13) that the only minimum of the function $G_1(a)$ is attained at the a that satisfies the equation

$$\frac{2}{\pi}\log\frac{1+a}{1-a} + T\left(\frac{2a}{a^2+1}\right) = \theta_\infty. \tag{3.14}$$

From (3.14) and (3.9) we deduce the global estimate

$$J_1(\nu, a) \geqslant f_1(\theta_\infty) = \frac{4a\mathrm{e}}{\pi(1+a^2)} \log\frac{1+a}{1-a}, \tag{3.15}$$

where a is the root of equation (3.14).

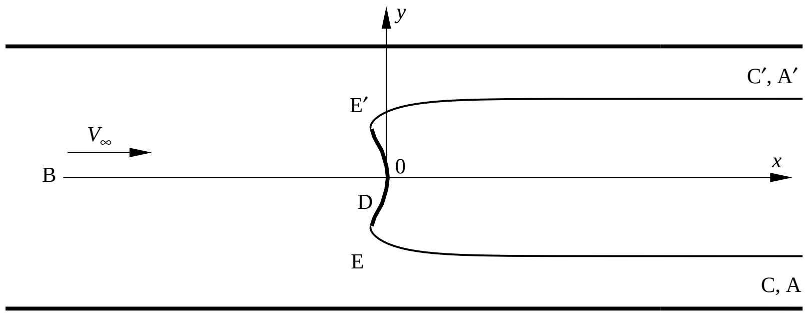

FIGURE 2. The 'deflector' inside the infinite channel.

Comparing (3.12) and (3.14) we come to the conclusion that $C = 1$ and the equality in (3.15) is possible if and only if

$$\nu(\sigma) = 1 - \log \cos \sigma. \tag{3.16}$$

Thus, the function (3.16) and the parameter a that satisfies (3.14) give the solution to the problem of the optimum deflector in the free jet. The minimum value of L/H for this case is $L_{\min 1}/H = f_1(\theta_\infty)$.

4. Optimum 'deflector' inside an infinite channel

The flow region for this particular case is shown in figure 2. The points C and A coincide, $\theta_\infty = 0$, $a = 0$. The word deflector in the title of the section is in quotes because for this case the deflector does not deflect the jet, but only contracts it. It follows from (2.9)–(2.11) that we need to minimize the functional

$$J_2(\nu, b) = \frac{4b^2}{\pi} \int_0^{\pi/2} \frac{\sin 2\sigma \exp[\nu(\sigma)]\, \mathrm{d}\sigma}{b^4 + 1 + 2b^2 \cos 2\sigma} \quad (\min) \tag{4.1}$$

under the restriction (2.10), because $L/H = kJ_2(\nu, b)$ at $a = 0$. The problems (3.1), (2.9) and (4.1), (2.10) are very similar and are solved by the same method. We again make the change (3.2) of the function to be found and apply Jensen's inequality (3.8) to the denominator of the transformed functional (4.1). Taking into account the transformed restriction (2.10) we arrive at the following estimate:

$$J_2(\nu, b) \geqslant G_2(b), \tag{4.2}$$

where

$$G_2(b) = \frac{8b^2}{\pi} g_2(b) \exp\left\{ -\frac{\pi}{4g_2(b)} \frac{\log k + T_1[2b/(1-b^2)]}{b(1-b^2)} \right\}, \tag{4.3}$$

$$g_2(b) = \int_0^{\pi/2} \frac{\sin \sigma\, \mathrm{d}\sigma}{b^4 + 1 + 2b^2 \cos 2\sigma} = \frac{1}{2(1-b^2)b} \arctan \frac{2b}{1-b^2}.$$

$$T_1(\beta) = -\mathrm{i}\, T(\mathrm{i}\, \beta) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\beta^{2n+1}}{(2n+1)^2}.$$

If in (4.3) the module of the argument of the function $T_1(\beta)$ is greater than unity ($|\beta| > 1$), then we make use of the formula $T_1(\beta) = T_1(1/\beta) + \log \beta$, which follows from (3.7). The equality in (4.2) is possible if and only if $\nu(\sigma)$ is defined by (3.11) with

$$C = -\frac{\pi}{2}\,\frac{\log k + T_1[2b/(1-b^2)]}{\arctan[2b/(1-b^2)]}. \tag{4.4}$$

Equation (4.4) follows from (3.11) and (2.10). Differentiation of the function (4.3) gives

$$\frac{\mathrm{d}G_2(b)}{\mathrm{d}b} = \frac{\pi G_2(b)}{1+b^2}\,\frac{\dfrac{2}{\pi}\arctan\dfrac{2b}{1-b^2} + T_1\left(\dfrac{2b}{1-b^2}\right) + \log k}{\arctan^2[2b/(1-b^2)]}. \tag{4.5}$$

From (4.3) and (4.5) we deduce that the only minimum of the function $G_2(b)$ with respect to $0 < b < 1$ is attained at the b which is the root of the equation

$$\frac{2}{\pi}\arctan\frac{2b}{1-b^2} + T_1\left(\frac{2b}{1-b^2}\right) = -\log k. \tag{4.6}$$

The global estimate for L/H follows from (4.2), (4.3) and (4.6):

$$L/H = kJ_2(\nu, b) \geqslant f_2(k) = \frac{4bk\mathrm{e}}{\pi(1-b^2)}\arctan\frac{2b}{1-b^2}, \tag{4.7}$$

where b satisfies (4.6).

From (4.4) and (4.6) we find that as in the previous section $C=1$. This means that equations (3.16) and (4.6) give the solution to the problem of the optimum deflector inside an infinite channel. The minimum value of L/H for this case is $L_{\min 2}/H = f_2(k)$.

5. General case

5.1. *Finding the global minimum*

The investigation of the general problem on minimizing the functional (2.11) under the restrictions (2.9), (2.10) is based on two facts. The first one is that for the particular solutions obtained in two previous sections the function $\nu(\sigma)$ is the same and defined by (3.16). The second one is that

$$J(\nu, a, b) = k[J_1(\nu, a) + J_2(\nu, b)]. \tag{5.1}$$

This important equation follows from (2.11) immediately. From (5.1), (3.15) and (4.7) we conclude that

$$J(\nu, a, b) \geqslant kf_1(\theta_\infty) + f_2(k) = L_{\min}/H, \tag{5.2}$$

the equality in (5.2) being possible if and only if the parameters a, b are the roots of equations (3.14), (4.6) correspondingly and $\nu(\sigma)$ is defined by (3.16). Thus, we have found the global minimum of L/H for the general problem (2.9)–(2.11). As one can see this minimum turns out to be the linear combination of two particular minima

$$L_{\min}/H = kL_{\min 1}/H + L_{\min 2}/H,$$

where the $L_{\min 1}/H = f_1(\theta_\infty)$ and $L_{\min 2}/H = f_2(k)$ are the minimum values of L/H for the deflector in a free jet and inside an infinite channel respectively. The graphs of the functions $f_1(\theta_\infty)$ and $f_2(k)$ are shown in figures 3 and 4.

5.2. *Dual problem*

As one can see from figure 3 the function $f_1(\theta_\infty)$ increases monotonically. It follows from this that the function $L_{\min}/H = kf_1(\theta_\infty) + f_2(k)$ for any fixed k increases

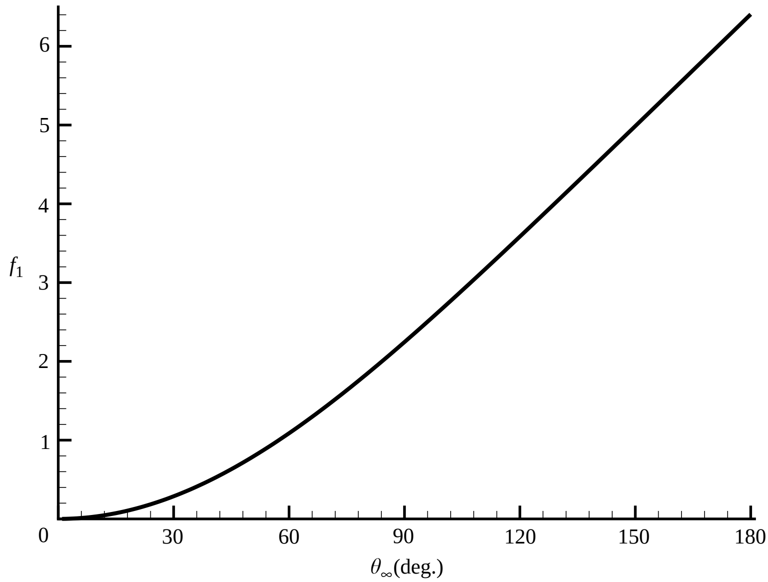

FIGURE 3. The function $L_{\min 1}/H = f_1(\theta_\infty)$ for the deflector in a free jet.

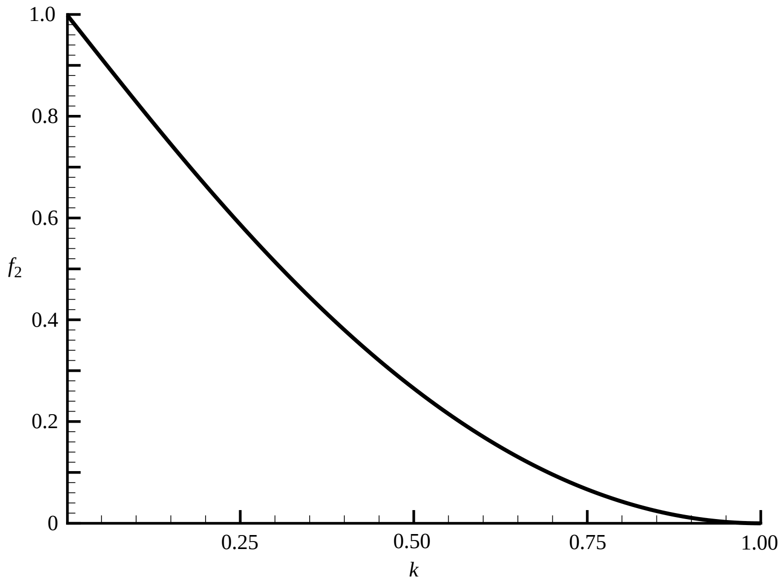

FIGURE 4. The function $L_{\min 2}/H = f_2(k)$ for the deflector inside an infinite channel.

monotonically too with an increase of θ_∞. This means that the solution (3.14), (3.16), (4.6) is also the solution to the dual problem of finding the optimum deflector shape that provides the maximum deflection angle θ_∞ under the given arclength $2L$ of the deflector. For this dual formulation we should find the parameter a from the equation

$$\frac{4a\mathrm{e}}{\pi(1+a^2)} \log \frac{1+a}{1-a} = [L/H - f_2(k)]/k, \tag{5.3}$$

that follows from (3.15) and (5.2). The maximum deflection angle is determined by (3.14).

But we should be careful in specifying L/H. Indeed, we cannot specify $L/H < f_2(k)$ because the minimum value of $f_1(\theta_\infty) = 0$ at $\theta_\infty = 0$, and as follows from (5.2) $f_2(k)$ is the minimal possible value of L/H that provides the given contraction coefficient k. On the other hand the inclination angle θ_∞ cannot be greater than π. From (3.14) and (5.2) we conclude that $\theta_\infty \leqslant \pi$ if

$$L/H \leqslant kP + f_2(k), \quad P = \frac{4a_* \mathrm{e}}{\pi(1+a_*^2)} \log \frac{1+a_*}{1-a_*},$$

where a_* is the root of the equation (3.14) with $\theta_\infty = \pi$. Numerical computations show that $a_* = 0.952036$ and $P = 6.405850$. The quantity $kP + f_2(k)$ is the minimum value of L/H that provides the revolution of the jet through an angle of 180°.

Thus for $L/H < f_2(k)$ the dual problem has no solution. In the range

$$f_2(k) \leqslant L/H \leqslant kP + f_2(k)$$

the dual problem has the unique solution defined by equations (3.16), (4.6) and (5.3). The case of long deflectors with $L/H > kP + f_2(k)$ is not interesting from the practical point of view and we do not consider it.

5.3. *Properties of optimum deflectors*

Inserting (3.16) into (2.3) gives rise to the equation

$$\omega(t) = \frac{2\mathrm{i}}{\pi} \log \frac{1-t}{1+t} - \mathrm{i}\,T(\alpha), \quad \alpha = \frac{2t}{t^2+1}. \tag{5.4}$$

So, the optimum function $\omega(t)$ turns out to be independent of θ_∞ and k. Knowing the function $\omega(t)$ and the mathematical parameters a, b we can compute all features of the flow over the optimum deflector. The conformal mapping of the parametric domain onto the flow region can be written as

$$z = kz_1(t) + z_2(t), \tag{5.5}$$

where

$$z_1(t) = \frac{2Ha^2}{\pi} \int_i^t \frac{(t^2-1)\exp[\omega(t)]\,\mathrm{d}t}{t(t^2-a^2)(1-a^2t^2)}, \quad z_2(t) = \frac{2Hb^2k}{\pi} \int_i^t \frac{(t^2-1)\exp[\omega(t)]\,\mathrm{d}t}{t(t^2+b^2)(1+b^2t^2)},$$

$z_1(t)$ and $z_2(t)$ being the conformal mappings of the parametric domain onto the flow regions for the optimum deflector in a free jet and inside an infinite channel respectively. Thus, the solution to the general problem is the linear combination of the two particular ones.

Making use of the formula (2.8) we determine the distance d from the stagnation point D to the end section CC′ of the channel:

$$\frac{d}{H} = \frac{2}{\pi} k(a^2+b^2)(1+a^2b^2) \times \int_0^1 \frac{\eta(1-\eta^4)\exp\{T_1[2\eta/(1-\eta^2)] + (4\arctan\eta)/\pi\}\,\mathrm{d}\eta}{(\eta^2+a^2)(1+a^2\eta^2)(\eta^2-b^2)(1-b^2\eta^2)}.$$

Here the integrand has a singularity at the point $\eta = b$ and should be computed as a Cauchy principal value. It is worthwhile to note that the distance d cannot be represented as a linear combination of the 'particular' d because for the particular case of a free jet $d = +\infty$ and for the particular case of an infinite channel $d = -\infty$.

Making use of (3.7) we find that

$$\omega(\mathrm{e}^{\mathrm{i}\sigma}) = -\log\frac{V(\sigma)}{V_0} + \mathrm{i}\,\Theta(\sigma),$$

where

$$\frac{V(\sigma)}{V_0} = \mathrm{e}^{-1}\cos\sigma$$

is the distribution of the flow velocity along the optimum deflector, and

$$\theta = \Theta(\sigma) = -\frac{\pi}{2} - \frac{1}{\pi}\log\frac{1+\cos\sigma}{1-\cos\sigma} + T(\cos\sigma) \tag{5.6}$$

is the distribution of the inclination of the velocity vector. The connection between V and θ along the optimum deflector is defined as

$$\theta = -\frac{\pi}{2} - \frac{1}{\pi}\log\frac{1+\mathrm{e}V/V_0}{1-\mathrm{e}V/V_0} + T\left(\mathrm{e}\frac{V}{V_0}\right), \quad 0 \leqslant \frac{V}{V_0} < \mathrm{e}^{-1}. \tag{5.7}$$

The shape of the optimum deflector is determined by the parametric equation

$$\left.\begin{aligned} x/H &= kx_1(\sigma) + x_2(\sigma), \\ y/H &= ky_1(\sigma) + y_2(\sigma), \end{aligned}\right\} \tag{5.8}$$

where

$$\left.\begin{aligned} x/H &= x_1(\sigma), \\ y/H &= y_1(\sigma) \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x/H &= x_2(\sigma), \\ y/H &= y_2(\sigma) \end{aligned}\right\} \tag{5.9}$$

are the parametric equations of the optimum deflector in a free jet and inside an infinite channel respectively, and

$$x_1(\sigma) = \frac{8a^2\mathrm{e}}{\pi}\int_\sigma^{\pi/2}\frac{\sin\gamma\cos\Theta(\gamma)\,\mathrm{d}\gamma}{a^4 - 2a^2\cos 2\gamma + 1}, \quad y_1(\sigma) = \frac{8a^2\mathrm{e}}{\pi}\int_\sigma^{\pi/2}\frac{\sin\gamma\sin\Theta(\gamma)\,\mathrm{d}\gamma}{a^4 - 2a^2\cos 2\gamma + 1},$$

$$x_2(\sigma) = \frac{8b^2k\mathrm{e}}{\pi}\int_\sigma^{\pi/2}\frac{\sin\gamma\cos\Theta(\gamma)\,\mathrm{d}\gamma}{b^4 + 2b^2\cos 2\gamma + 1}, \quad y_2(\sigma) = \frac{8b^2k\mathrm{e}}{\pi}\int_\sigma^{\pi/2}\frac{\sin\gamma\sin\Theta(\gamma)\,\mathrm{d}\gamma}{b^4 + 2b^2\cos 2\gamma + 1}.$$

Let s be the arc coordinate of deflector points measured clockwise from the stagnation point D. From (2.8) and (5.4) after a little algebra we deduce

$$\frac{s}{H} = \frac{2k\mathrm{e}}{\pi}\left[\frac{a}{a^2+1}\log\frac{a^2+1+2a\cos\sigma}{a^2+1-2a\cos\sigma} + \frac{2b}{1-b^2}\arctan\frac{2b\cos\sigma}{1-b^2}\right]. \tag{5.10}$$

As follows from equations (5.6) and (5.10) the inclination of the velocity vector $\theta \to \infty$ as $s \to L$ ($\sigma \to 0$). Therefore the optimal deflectors take the form of spirals in the vicinity of the end point E. The reason is that the velocity distribution along the boundary of the flow has a discontinuity at the point of separation E. Indeed, on the free streamline EA we have $V = V_0$, whereas at the end point E on the optimum deflector $V = \mathrm{e}^{-1}V_0$. From (5.6) and (5.10) it is easy to see that in the vicinity of the point E (as $\sigma \to 0$) the following asymptotic relations hold:

$$\frac{L-s}{H} = R(1-\cos\sigma) + o(1-\cos\sigma), \tag{5.11}$$

$$\theta = -\frac{\pi}{2} + \frac{1}{\pi}\log(1-\cos\sigma)/2 + O(1-\cos\sigma), \tag{5.12}$$

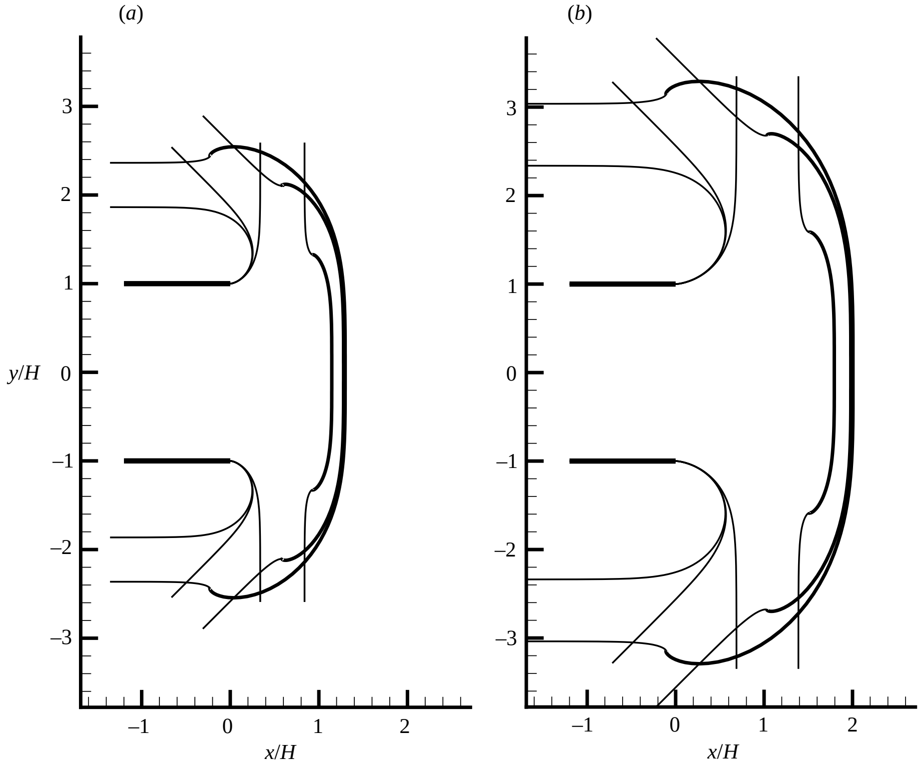

FIGURE 5. Flow regions for the optimum deflectors at (a) $k = 0.5$, (b) $k = 0.7$ and $\theta_\infty = 90°, 135°, 180°$.

where

$$R = \frac{8ek}{\pi}\left[\frac{a^2}{(1-a^2)^2} + \frac{b^2}{(1-b^2)^2}\right],$$

and in deducing (5.12) we need to take into account that $T(1) = \pi/4$. We exclude $(1 - \cos\sigma)$ from (5.11), (5.12) to write

$$\theta = -\frac{\pi}{4} - \frac{1}{\pi}\log 2R + \frac{1}{\pi}\log\frac{L-s}{H} + O\left(\frac{L-s}{H}\right).$$

From this relation it is possible to deduce that the spiral at the end point E is asymptotically logarithmic and is described by the equation

$$r = 2R\mathrm{e}^{M+\pi\phi}, \tag{5.13}$$

where $M = -\pi^2/4 - \pi\arctan\pi + \ln(\pi/\sqrt{\pi^2+1})$, ϕ is a polar angle (negative for the point E), and r is the distance between a point on the optimal deflector and the pole E of the spiral.

The flow near the point E is asymptotically close to a spiral flow between two free streamlines with constant but different velocities on each of them. Such flows are discussed in Birkhoff & Zarantonello (1957, figure 35). It is to be noted that the spirals near the end points E and E′ cannot be seen at any scales. Indeed, it follows from equation (5.13) that with every half-revolution of the radius-vector $\boldsymbol{r}$ its length decreases by the factor $e^{-\pi^2} \approx 0.5 \times 10^{-4}$. Therefore, if we plot a part of the spiral corresponding to one half-revolution of the radius vector $\boldsymbol{r}$ the remaining part will be like a point at any scales. As was demonstrated in Maklakov (1999) and Maklakov (1997) the contribution of these tiny spirals to the hydrodynamic forces is negligible.

In figure 5(a, b) we show the flow regions for the optimum deflectors. For convenience of comparison the y-axis does not go through the point D but through the end section CC′ of the channel. As one can see from figure 5 with an increase of the deflection angle and the width of the jet, the minimum arclength $2L$ and the distance d from the deflector to the channel increase too.

6. Conclusions

We have found an exact analytical solution to the problem of the deflector of optimum shape which divides the jet that effuses from a semi-infinite channel. First we have investigated two particular problems: on the optimum deflector in a free jet and inside an infinite channel. In so doing we have estimated the minimized functionals by means of Jensen's inequality, the estimates turning out to be the functions of only one variable. We have proven that the minima of these functions coincide with those of the initial functionals. Thus both particular problems have been reduced to minimization of the functions of one variable. This allows us to demonstrate that the minima obtained are global. After solving the particular problems we have proven a surprising fact: the solution to the general nonlinear problem is the linear combination of the particular ones, both particular problems being also nonlinear.

The work was supported by the Russian Foundation of Basic Research under grants No 05-01-00794, No 03-01-00015.

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